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Mathematical Preliminaries

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Mathematical Preliminaries

- Sets
- Functions
- Relations
- Graphs
- Proof Techniques

SETS

A set is a collection of elements

$$A = \{1, 2, 3\}$$

$$B = \{train, bus, bicycle, airplane\}$$

We write

$$1 \in A$$

$$ship \notin B$$

SETS

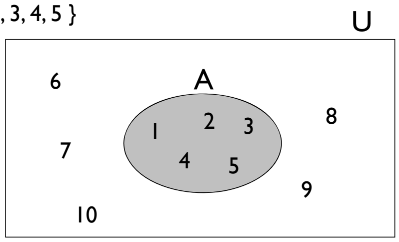
Set Representations

C = { a, b, c, d, e, f, g, h, i, j, k }
C = { a, b, …, k } ⟶ finite set

S = { 2, 4, 6, … } ⟶ infinite set
S = { j : j > 0, and j = 2k for some k>0 }
S = { j : j is nonnegative and even }

SETS

A = { 1, 2, 3, 4, 5 }



Universal Set: all possible elements
U = { 1 , … , 10 }

SETS

Set Operations:

A = { 1, 2, 3 } B = { 2, 3, 4, 5}

- Union

  A U B = { 1, 2, 3, 4, 5 }

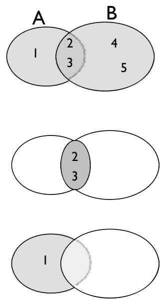
- Intersection

  A ∩ B = { 2, 3 }

- Difference

  A - B = { 1 }

  B - A = { 4, 5 }



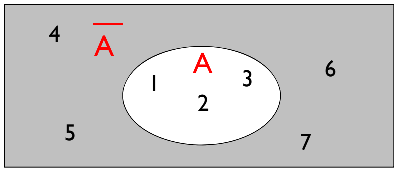
Venn diagrams

SETS

Complement:

Universal set = $\{1, \ldots, 7\}$

$A = \{1, 2, 3\} \Rightarrow \overline{A} = \{4, 5, 6, 7\}$

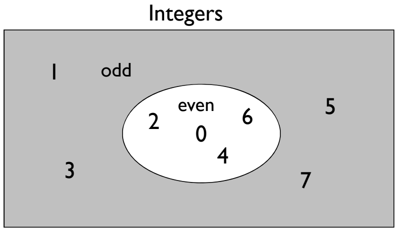


$\overline{\overline{A}} = A$

SETS

Complement:

$\overline{\{\text{ even integers }\}} = \{\text{ odd integers }\}$



SETS

DeMorgan's Laws

$\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

SETS

Empty, Null Set ($\emptyset$):

$\emptyset = \{\}$

$S \cup \emptyset = S$

$S \cap \emptyset = \emptyset$

$S - \emptyset = S$

$\emptyset - S = \emptyset$

$\overline{\emptyset}$ = Universal Set

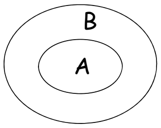
SETS

Subset:

$A = \{1, 2, 3\}$ $B = \{1, 2, 3, 4, 5\}$

$A \subseteq B$

Proper Subset: $A \subset B$

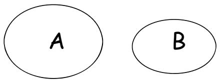


SETS

Disjoint Sets:

$A = \{1, 2, 3\}$ $B = \{5, 6\}$

$A \cap B = \emptyset$



SETS

Set Cardinality:

For finite sets

A = { 2, 5, 7 }

|A| = 3

SETS

Powerset of a set:

A powerset is a set of sets

S = { a, b, c }

Powerset of S = the set of all the subsets of S

2^S = { ∅, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c} }

Observation: $| 2^S | = 2^{|S|}$ ($8 = 2^3$)

SETS

Cartesian Product:

A = { 2, 4 }   B = { 2, 3, 5 }

A X B = { (2, 2), (2, 3), (2, 5), (4, 2), (4, 3), (4, 5) }

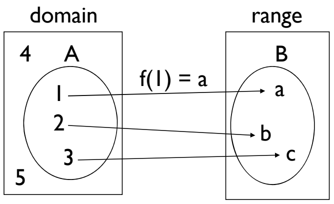
|A X B| = |A| |B|

A X B X … X Z

Generalizes to more than two sets

SETS

Functions:



f :A -> B

If A = domain, then f is a total function, otherwise f is a partial function

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RELATIONS

R = $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots\}$

x_i R y_i

e. g. if R = '>': 2 > 1, 3 > 2, 3 > 1

Equivalence Relations

- Reflexive: x R x
- Symmetric: x R y $\Rightarrow$ y R x
- Transitive: x R y and y R z $\Rightarrow$ x R z

Example: R = '='

- x = x
- x = y $\Rightarrow$ y = x
- x = y and y = z $\Rightarrow$ x = z

Equivalence Classes

For equivalence relation R

equivalence class of x = {y : x R y}

Example:

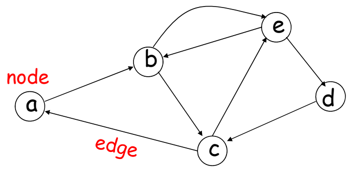
R = { (1, 1), (2, 2), (1, 2), (2, 1),
(3, 3), (4, 4), (3, 4), (4, 3) }

Equivalence class of 1 = {1, 2}

Equivalence class of 3 = {3, 4}

GRAPHS

A directed graph



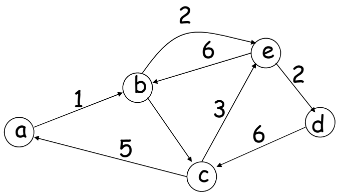
- Nodes (Vertices)

V = { a, b, c, d, e }

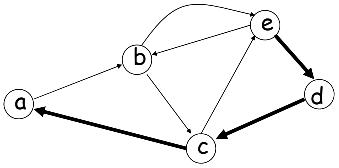
- Edges

E = { (a,b), (b,c), (b,e),(c,a), (c,e), (d,c), (e,b), (e,d) }

Labeled Graph



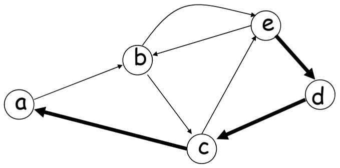
Walk



Walk is a sequence of adjacent edges

(e, d), (d, c), (c, a)

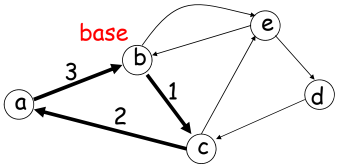
Path



Path is a walk where no edge is repeated

Simple path: no node is repeated

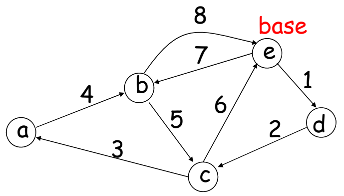
Cycle



Cycle: a walk from a node (base) to itself

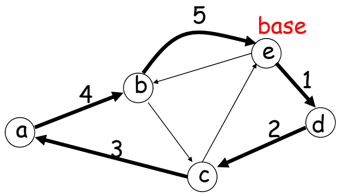
Simple cycle: only the base node is repeated

Euler Tour



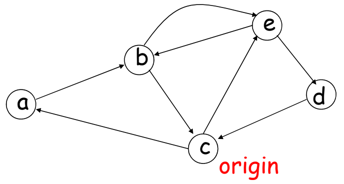
A cycle that contains each edge once

Hamiltonian Cycle

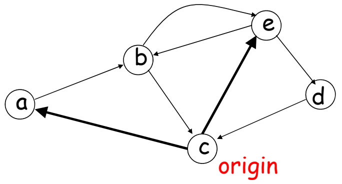


A simple cycle that contains all nodes

Finding All Simple Paths



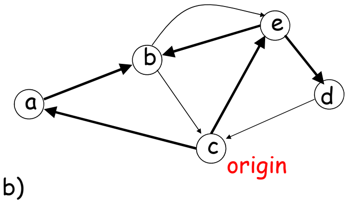
Step 1



(c, a)

(c, e)

Step 2



(c, a)

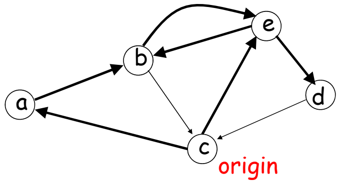
(c, a), (a, b)

(c, e)

(c, e), (e, b)

(c, e), (e, d)

Step 3



(c, a)

(c, a), (a, b)

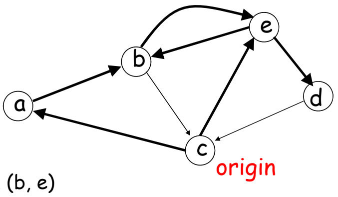
(c, a), (a, b), (b, e)

(c, e)

(c, e), (e, b)

(c, e), (e, d)

Step 4



(c, a)

(c, a), (a, b)

(c, a), (a, b), (b, e)

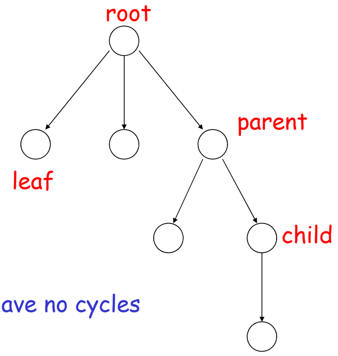
(c, a), (a, b), (b, e), (e,d)

(c, e)

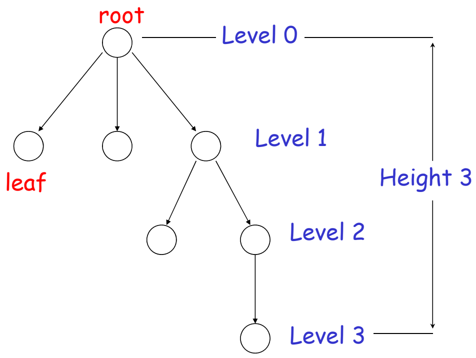
(c, e), (e, b)

(c, e), (e, d)

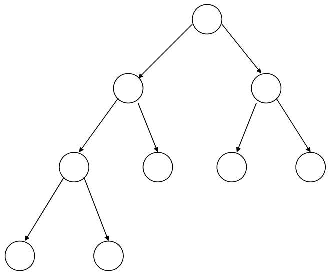
Trees



Trees have no cycles



Binary Trees



PROOF TECHNIQUES

- Proof by induction
- Proof by contradiction

Induction

We have statements $P_1, P_2, P_3, \ldots$

If we know

- for some b that $P_1, P_2, \ldots, P_b$ are true
- for any $k >= b$ that

$P_1, P_2, \ldots, P_k$ imply P_{k+1}

Then

Every P_i is true

Proof by Induction

- Inductive basis

  Find $P_1, P_2, \dots, P_b$ which are true

- Inductive hypothesis

  Let's assume $P_1, P_2, \dots, P_k$ are true, for any k >= b

- Inductive step

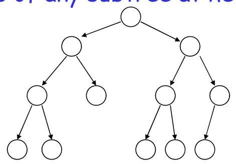
  Show that P_{k+1} is true

Example

Theorem: A binary tree of height n has at most 2^n leaves.

Proof by induction:

let L(i) be the maximum number of leaves of any subtree at height i



We want to show: L(i) <= 2^i

- Inductive basis

  L(0) = 1 (the root node)

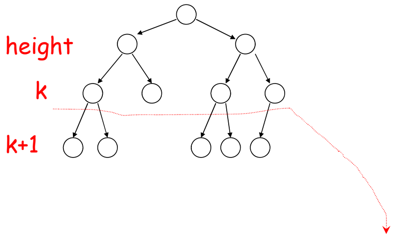
- Inductive hypothesis

  Let's assume L(i) <= 2^i for all i = 0, 1, …, k

- Induction step

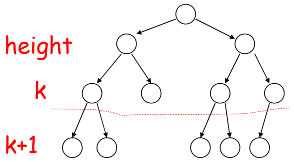
  we need to show that L(k + 1) <= 2^{k+1}

Induction Step



From Inductive hypothesis: L(k) <= 2^k

Induction Step



L(k) <= 2^k

L(k+1) <= 2 * L(k) <= 2 * 2^k = 2^{k+1}

(we add at most two nodes for every leaf of level k)

Remark

Recursion is another thing

Example of recursive function:

f(n) = f(n-1) + f(n-2)

f(0) = 1, f(1) = 1

Proof by Contradiction

We want to prove that a statement P is true

- we assume that P is false
- then we arrive at an incorrect conclusion
- therefore, statement P must be true

Example

Theorem: $\sqrt{2}$ is not rational

Proof:

Assume by contradiction that it is rational

$\sqrt{2}$ = n/m

n and m have no common factors

We will show that this is impossible

$\sqrt{2}$ = n/m $\Rightarrow$ $2\ m^2 = n^2$

Therefore, n^2 is even $\Rightarrow$ n is even

n = 2 k

$2\ m^2 = 4k^2$ $\Rightarrow$ $m^2 = 2k^2$ $\Rightarrow$ m is even

m = 2 p

Thus, m and n have common factor 2

Contradiction!

Languages

A language is a set of strings

String: A sequence of letters

Examples: “cat”, “dog”, “house”, …

Defined over an alphabet:

$$\Sigma = \{a, b, c, \ldots, z\}$$

Alphabets and Strings

We will use small alphabets: $\Sigma = \{a, b\}$

Strings

a

ab $\qquad u = ab$

$abba$ $\qquad v = bbbaaa$

$baba$ $\qquad w = abba$

$aaabbbaabab$

String Operations

$w = a_1 a_2 \cdots a_n$ $\quad$ $abba$

$v = b_1 b_2 \cdots b_m$ $\quad$ $bbbaaa$

Concatenation

$wv = a_1 a_2 \cdots a_n b_1 b_2 \cdots b_m$ $\quad$ $abbabbbaaa$

$w = a_1 a_2 \cdots a_n$ $\quad$ $ababaaabbb$

Reverse

$w^R = a_n \cdots a_2 a_1$ $\quad$ $bbbaaababa$

String Length

$w = a_1 a_2 \cdots a_n$

Length: $|w| = n$

Examples: $|abba| = 4$

$|aa| = 2$

$|a| = 1$

Length of Concatenation

$|uv| = |u| + |v|$

Example: $u = aab, \quad |u| = 3$

$v = abaab, \quad |v| = 5$

$|uv| = |aababaab| = 8$

$|uv| = |u| + |v| = 3 + 5 = 8$

Empty String

A string with no letters: λ

Observations: $|\lambda| = 0$

$\lambda w = w\lambda = w$

$\lambda abba = abba\lambda = abba$

Substring

Substring of string:

a subsequence of consecutive characters

String	Substring
$\underline{ab}bab$	ab
$\underline{abba}b$	$abba$
$ab\underline{b}ab$	b
$a\underline{bbab}$	$bbab$

Prefix and Suffix

$abbab$

Prefixes	Suffixes
λ	$abbab$
a	$bbab$
ab	bab
abb	ab
$abba$	b
$abbab$	λ

$w = uv$

prefix

suffix

Another Operation

$$w^n = \underbrace{ww\cdots w}_{n}$$

Example: $(abba)^2 = abbaabba$

Definition: $w^0 = \lambda$

$(abba)^0 = \lambda$

The * Operation

$\Sigma *$: the set of all possible strings from alphabet Σ

$\Sigma = \{a, b\}$

$\Sigma * = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$

The + Operation

Σ^+ : the set of all possible strings from alphabet Σ except λ

$\Sigma = \{a, b\}$

$\Sigma * = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$

$\Sigma^+ = \Sigma * - \lambda$

$\Sigma^+ = \{a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$

Languages

A language is any subset of $\Sigma *$

Example: $\Sigma = \{a, b\}$

$\Sigma * = \{\lambda, a, b, aa, ab, ba, bb, aaa, \ldots\}$

Languages: $\{\lambda\}$

$\{a, aa, aab\}$

$\{\lambda, abba, baba, aa, ab, aaaaaa\}$

Note that:

Sets $\varnothing = \{\ \} \neq \{\lambda\}$

Set size $|\{\ \}| = |\varnothing| = 0$

Set size $|\{\lambda\}| = 1$

String length $|\lambda| = 0$